CYCU-HEP-15-01
EPHOU-15-002

Gauge extension of non-Abelian discrete flavor symmetry

Florian Beye[1*], Tatsuo Kobayashi[2†] and Shogo Kuwakino[3‡]

[1] *Department of Physics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya 464-8602, Japan*
[2] *Department of Physics, Hokkaido University, Sapporo 060-0810, Japan*
[3] *Department of Physics, Chung-Yuan Christian University, 200, Chung-Pei Rd. Chung-Li,320, Taiwan*

Abstract

We investigate a gauge theory realization of non-Abelian discrete flavor symmetries and apply the gauge enhancement mechanism in heterotic orbifold models to field-theoretical model building. Several phenomenologically interesting non-Abelian discrete symmetries are realized effectively from a $U(1)$ gauge theory with a permutation symmetry. We also construct a concrete model for the lepton sector based on a $U(1)^2 \rtimes S_3$ symmetry.

*Electronic address: fbeye@eken.phys.nagoya-u.ac.jp
†Electronic address: kobayashi@particle.sci.hokudai.ac.jp
‡Electronic address: kuwakino@cycu.edu.tw

1 Introduction

The flavor structure of quarks and leptons in the standard model is mysterious. Why are there three generations? Why are their masses hierarchically different from each other? Why do they show the specific mixing angles? It is challenging to try to solve this flavor mystery. A flavor symmetry could play an important role in particle physics models in order to understand the flavor structure of quarks and leptons. Since the Yukawa matrices of the standard model include many parameters, flavor symmetries are useful to effectively reduce the number of parameters and to obtain some predictions for experiments. In particular, non-Abelian discrete flavor symmetries can be key ingredients to make models with a suitable flavor structure. Indeed, there are many works of flavor models utilizing various non-Abelian discrete flavor symmetries (see [1, 2, 3] for reviews).

It is known that some non-Abelian discrete flavor symmetries have a stringy origin. In particular, in orbifold compactification of heterotic string theory [4, 5, 6, 7, 8, 9, 10, 11, 12] (also see a review [13]), non-Abelian discrete symmetries D_4 and $\Delta(54)$ respectively arise from one- and two-dimensional orbifolds, S_1/Z_2 and T_2/Z_3, as discussed in [14][1]. The non-Abelian discrete symmetries originate from a geometrical property of extra-dimensional orbifolds, the permutation symmetry of orbifold fixed points, and a string selection rule between closed strings. Phenomenological applications of string derived non-Abelian discrete symmetries to flavor models are analyzed, e.g. in [19].

Furthermore, in [20], it is argued that the non-Abelian discrete symmetries D_4 and $\Delta(54)$ have a gauge origin within the heterotic string theory. Namely, these symmetries are respectively enhanced to continuous gauge symmetries $U(1) \rtimes Z_2$ and $U(1)^2 \rtimes S_3$ at a symmetry enhancement point in the moduli space of orbifolds. After certain scalar fields which are associated with the Kähler moduli fields get vacuum expectation values, the $U(1)$ symmetries break down to Abelian discrete subgroups, and there remains a $Z_4 \rtimes Z_2 \cong D_4$ or $(Z_3 \times Z_3) \rtimes S_3 \cong \Delta(54)$ symmetry group, respectively. This result suggests that a non-Abelian discrete symmetry can be regarded as a remnant of a continuous gauge symmetry. Also, this result could provide us with a new insight on model building for flavor physics.

Various non-Abelian discrete symmetries other than D_4 and $\Delta(54)$ have been used in field-theoretical model building, e.g. S_3, S_4, A_4, $\Delta(3N^2)$, $\Delta(6N^2)$ (see [1, 2, 3]). Thus, it is important to extend the stringy derivation of D_4 and $\Delta(54)$ from $U(1) \rtimes Z_2$ and $U(1)^2 \rtimes S_3$, by studying a field-theoretical derivation of other non-Abelian discrete flavor symmetries from $U(1)^m \rtimes S_n$ or $U(1)^m \rtimes Z_n$ (See also [21]). That is the purpose of this paper. Some of them may be reproduced from other types of string compactifications.

In this paper we consider an extension of the argument of the gauge origin in [20] to field-theoretical model building. We show that phenomenologically interesting non-Abelian discrete symmetries can be embedded into $U(1)^m \rtimes S_n$ or $U(1)^m \rtimes Z_n$ continuous gauge theory. Spontaneous symmetry breaking of $U(1)^m$ to Abelian discrete symmetries leads to non-Abelian

[1]Similar non-Abelian discrete symmetries including $\Delta(27)$ can appear in intersecting/magnetized D-brane models [15, 16, 17]. See also [18].

discrete flavor symmetries. In the next section we discuss a gauge theory realization of non-Abelian discrete symmetries. In section 3, we show a concrete lepton flavor model based on a $U(1)$ flavor symmetry. Section 4 is devoted to conclusions.

2 Gauge extension of non-Abelian discrete symmetry

In this section we investigate a field theoretical model building technique in which non-Abelian discrete symmetries have a continuous gauge symmetry origin. We start with a gauge theory with group structure of the form $U(1)^n \rtimes S_m$ or $U(1)^n \rtimes Z_m$. Then, by giving a suitable VEV to a scalar field, a non-Abelian discrete symmetry is realized effectively.

2.1 S_3 group

We consider a $U(1) \rtimes Z_2$ model with the field contents as in Table 1. The action of the Z_2 symmetry on the $U(1)$ charge q is given by

$$Z_2 : q \to -q. \tag{1}$$

By this we mean that the $U(1)$ gauge field A_μ transforms as $A_\mu \to -A_\mu$, and that the oppositely charged fields in this model transform into each other, e.g. $U_1 \leftrightarrow U_2$ and $M_1 \leftrightarrow M_2$. This implies that the kinetic (and gauge interaction) terms are invariant under the Z_2.

Now, we consider VEVs for fields U_i obeying the relation

$$\langle U_1 \rangle = \langle U_2 \rangle. \tag{2}$$

This VEV relation maintains the original Z_2 permutation symmetry,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3}$$

but breaks the $U(1)$ group to a discrete Z_3 subgroup since the field M_i has $U(1)$ charge $\pm 1/3$. The Z_3 charges are 1 for the field M_1 and 2 for the field M_2, so the Z_3 action is expressed by

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \tag{4}$$

with the cubic root $\omega = e^{2\pi i/3}$. The combination of the two actions (3) and (4) gives rise to a non-Abelian discrete symmetry, which is nothing but $S_3 \cong Z_3 \rtimes Z_2$. It turns out that (M_1, M_2) forms a doublet of this S_3 group.

Next, we read off the S_3 representation of the other matter fields. First, the field M can be regarded as the trivial singlet $\mathbf{1}$ of the S_3 group. In the case of (M'_1, M'_2), we see that these fields have trivial Z_3 charges. Then we can perform a change of basis as $\tilde{M}'_1 \equiv M'_1 + M'_2$ and $\tilde{M}'_2 \equiv M'_1 - M'_2$. In this basis, the Z_2 action is given by $\tilde{M}'_1 \to \tilde{M}'_1$ and $\tilde{M}'_2 \to -\tilde{M}'_2$. Hence, (M'_1, M'_2) forms a $\mathbf{1} \oplus \mathbf{1}'$ of the S_3 group. As a result, we can reproduce all irreducible representations of the S_3 group.

Field	$U(1)$ charge	Z_3 charge	S_3 rep.
U_1, U_2	$+1, -1$	$0, 0$	—
M_1, M_2	$+\frac{1}{3}, -\frac{1}{3}$	$1, 2$	$\mathbf{2}$
M	0	0	$\mathbf{1}$
M'_1, M'_2	$+1, -1$	$0, 0$	$\mathbf{1} \oplus \mathbf{1}'$

Table 1: Field contents of the $U(1) \rtimes Z_2$ model for the S_3 group. Besides the $U(1)$ charges, the charges under the unbroken discrete Z_3 subgroup of $U(1)$ are shown. Representations under the resulting S_3 group are also shown.

2.2 D_4 group

Now, we consider a $U(1) \rtimes Z_2$ model with the field contents as in Table 2. This model is based on a $U(1)$ symmetry and possesses an additional Z_2 symmetry which acts on the $U(1)$ charge as in the previous case (1), so the fields transform as $U_1 \leftrightarrow U_2$ and $M_1 \leftrightarrow M_2$ etc. We consider the following VEV relation

$$\langle U_1 \rangle = \langle U_2 \rangle. \tag{5}$$

This VEV relation maintains the original Z_2 permutation symmetry,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{6}$$

but breaks the $U(1)$ group to its discrete Z_4 subgroup. The Z_4 charges for M_1 and M_2 are 1 and 3 respectively, hence the Z_4 action is written as

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{7}$$

The combination of actions (6) and (7) leads to the non-Abelian discrete symmetry $D_4 \cong Z_4 \rtimes Z_2$. It turns out that (M_1, M_2) forms the doublet of the D_4 group.

Next, we read off the D_4 representation of the other matter fields. First, the field M can be regarded as the trivial singlet $\mathbf{1}_{++}$ of D_4. In the case of a set of fields (M'_1, M'_2), we make redefinitions as $\tilde{M}'_1 \equiv M'_1 + M'_2$ and $\tilde{M}'_2 \equiv M'_1 - M'_2$. In this basis, the Z_2 action acts as $\tilde{M}'_1 \to \tilde{M}'_1$ and $\tilde{M}'_2 \to -\tilde{M}'_2$. Thus, (M'_1, M'_2) forms a $\mathbf{1}_{++} \oplus \mathbf{1}_{--}$ of the D_4 group. For the fields (N_1, N_2), both fields have Z_4 charge 2. Then we can take a linear combination as $\tilde{N}_1 \equiv N_1 + N_2$ and $\tilde{N}_2 \equiv N_1 - N_2$, and observe that the Z_2 action acts as $\tilde{N}_1 \to \tilde{N}_1$ and $\tilde{N}_2 \to -\tilde{N}_2$. Then $(\tilde{N}_1, \tilde{N}_2)$ forms $\mathbf{1}_{+-} \oplus \mathbf{1}_{-+}$ of the D_4 group. As a result, we can reproduce all irreducible representations of the D_4 group by a suitable field setup.

2.3 S_4 group

We consider a $U(1)^2 \rtimes S_3$ model with the field contents as in Table 3. This model has a gauge $U(1)^2$ symmetry and fields are characterized by two $U(1)$ charges q_1 and q_2. We define the

Field	$U(1)$ charge	Z_4 charge	D_4 rep.
U_1, U_2	$+1, -1$	$0, 0$	—
M_1, M_2	$+\frac{1}{4}, -\frac{1}{4}$	$1, 3$	$\mathbf{2}$
M	0	0	$\mathbf{1}_{++}$
M_1', M_2'	$+1, -1$	$0, 0$	$\mathbf{1}_{++} \oplus \mathbf{1}_{--}$
N_1, N_2	$+\frac{1}{2}, -\frac{1}{2}$	$2, 2$	$\mathbf{1}_{+-} \oplus \mathbf{1}_{-+}$

Table 2: Field contents of the $U(1) \rtimes Z_2$ model for the D_4 group. Besides the $U(1)$ charges, the charges under the unbroken discrete Z_4 subgroup of $U(1)$ are shown. Representations under the resulting D_4 group are also shown.

two dimensional $U(1)^2$ charges e_1, e_2 and e_3 used in the table as

$$e_1 \equiv (\sqrt{2}, 0), \;\; e_2 \equiv (-\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}), \;\; e_3 \equiv (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{2}). \tag{8}$$

The additional non-Abelian discrete S_3 symmetry is generated by a 120 degree rotation and a reflection on the two-dimensional $U(1)^2$ charge plane (q_1, q_2) as

$$\text{Rotation}: \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \rightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{9}$$

$$\text{Reflection}: \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \tag{10}$$

The S_3 action permutes e_1, e_2 and e_3, which corresponds to a permutation of the fields as $U_1 \leftrightarrow U_2 \leftrightarrow U_3$ and $M_1 \leftrightarrow M_2 \leftrightarrow M_3$. We consider the VEV relation as

$$\langle U_1 \rangle = \langle U_2 \rangle = \langle U_3 \rangle. \tag{11}$$

This VEV relation maintains S_3,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{12}$$

but breaks the $U(1)^2$ group down to a discrete Z_2^2 subgroup. The Z_2 charges z_1 and z_2 in Table 3 are determined from the $U(1)^2$ charges as $z_1 = 2(q_1/\sqrt{2} - q_2/\sqrt{6})$ (mod 2) and $z_2 = 2(q_1/\sqrt{2} + q_2/\sqrt{6})$ (mod 2). Then, the Z_2^2 action is given by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{13}$$

The combination of (12) and (13) gives rise to the non-Abelian discrete symmetry $S_4 \cong (Z_2 \times Z_2) \rtimes S_3$. It turns out that (M_1, M_2, M_3) forms the triplet $\mathbf{3}$ of the S_4 group.

Field	$U(1)^2$ charge	Z_2^2 charge	S_4 rep.
U_1, U_2, U_3	$-e_1, -e_2, -e_3$	$(0,0),(0,0),(0,0)$	—
M_1, M_2, M_3	$\frac{e_1}{2}, \frac{e_2}{2}, \frac{e_3}{2}$	$(1,1),(1,0),(0,1)$	$\mathbf{3}$
M	0	$(0,0)$	$\mathbf{1}$
N_1, N_2, N_3	e_1, e_2, e_3	$(0,0),(0,0),(0,0)$	$\mathbf{1} \oplus \mathbf{2}$

Table 3: Field contents of the $U(1)^2 \rtimes S_3$ model for the S_4 group. Besides the $U(1)^2$ charges, the charges under the unbroken discrete Z_2^2 subgroup of $U(1)^2$ are shown. Representations under the resulting S_4 group are also shown.

Next, we read off the S_4 representation of the other matter fields. First, the field M can be regarded as the trivial singlet $\mathbf{1}$ of S_4. In the case of the fields (N_1, N_2, N_3), we make redefinitions as $\tilde{N}_1 \equiv (N_1 + N_2 + N_3)/\sqrt{3}$, $\tilde{N}_2 \equiv (N_1 + \omega N_2 + \omega^2 N_3)/\sqrt{3}$ and $\tilde{N}_3 \equiv (N_1 + \omega^2 N_2 + \omega N_3)/\sqrt{3}$. In this basis, the three fields transform as the $\mathbf{1} \oplus \mathbf{2}$ of the S_3 group. After the VEV, these fields have the trivial Z_2^2 charge $(0,0)$, so they correspond to $\mathbf{1} \oplus \mathbf{2}$ of S_4. Note, that fields with opposite $U(1)^2$ charges $-e_i/2$ have, after $U(1)^2$ breaking, the same Z_2^2 charges as the fields M_i. Hence, such fields also lead to the $\mathbf{3}$ of S_4. As a result, we can realize the $\mathbf{1}, \mathbf{1} \oplus \mathbf{2}, \mathbf{3}$ representations of the S_4 group in this setup.

We have introduced the specific combination of $U(1)^2$ charges, e_1, e_2, and e_3 which can be interpreted as weights of the fundamental $SU(3)$ triplet (or anti-triplet) representation. Then, the action of the S_3 group on the e_i corresponds to the action of the Weyl group of $SU(3)$ on the triplet weights. Thus, one might wonder about a $SU(3)$ origin of this setup. In fact, $U(1)^2 \rtimes S_3$ is a subgroup of $SU(3)$ where $U(1)^2$ furnishes maximal torus and S_3 is a lift of the Weyl group into $SU(3)$. Also, note that the representation matrices (12) of S_3 do not actually belong to $SU(3)$, so they give rise to genuine $U(1)^2 \rtimes S_3$ representations. The fundamental triplet and anti-triplet of $SU(3)$ also give rise to $U(1)^2 \rtimes S_3$ representations which we did not cover here (in these cases, the representation matrices are given by those in (12) amended by a minus sign). For a short remark on these kinds of representations please refer to the conclusion section.

Furthermore, in a stringy realization of $\Delta(54)$, the $SU(3)$ gauge symmetry appears in toroidal compactification at a symmetry enhanced point. Then, by a Z_3 orbifolding the charged root vectors are projected out [20], leaving a symmetry group $U(1)^2 \rtimes S_3$.

To realize $\Delta(54)$, A_4 and $\Delta(27)$ in the next subsections, we also use the vectors e_1, e_2 and e_3, as well as the Weyl reflections and the Coxeter elements.

2.4 $\Delta(54)$ group

We consider a $U(1)^2 \rtimes S_3$ model for the $\Delta(54)$ group, with field contents given as in Table 4. The difference from the previous subsection is that the matter fields now have relative $U(1)$ charges of 1/3 when compared to the fields U_i. Then, by the VEV relation (11) for the field U_i, the S_3 symmetry remains but $U(1)^2$ is broken down to its Abelian subgroup Z_3^2.

Field	$U(1)^2$ charge	Z_3^2 charge	$\Delta(54)$ rep.
U_1, U_2, U_3	$-e_1, -e_2, -e_3$	$(0,0), (0,0), (0,0)$	—
M_1, M_2, M_3	$\frac{e_1}{3}, \frac{e_2}{3}, \frac{e_3}{3}$	$(1,1), (2,0), (0,2)$	$\mathbf{3}_{1(1)}$
M_1', M_2', M_3'	$-\frac{e_1}{3}, -\frac{e_2}{3}, -\frac{e_3}{3}$	$(2,2), (1,0), (0,1)$	$\mathbf{3}_{1(2)}$
M	0	$(0,0)$	$\mathbf{1}_+$
N_1, N_2, N_3	e_1, e_2, e_3	$(0,0), (0,0), (0,0)$	$\mathbf{1}_+ \oplus \mathbf{2}_1$

Table 4: Field contents of the $U(1)^2 \rtimes S_3$ model for the $\Delta(54)$ group. Besides the $U(1)^2$ charges, the charges under the unbroken discrete Z_3^2 subgroup of $U(1)^2$ are shown. Representations under the resulting $\Delta(54)$ group are also shown.

The two Z_3 charges z_1, z_2 in Table 4 are determined as $z_1 = 3(q_1/\sqrt{2} - q_2/\sqrt{6})$ (mod 3) and $z_2 = 3(q_1/\sqrt{2} + q_2/\sqrt{6})$ (mod 3), and the Z_3^2 action is described by

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^{-1} \end{pmatrix}. \tag{14}$$

The actions (12) and (14) together generate the non-Abelian discrete symmetry $\Delta(54) \cong (Z_3 \times Z_3) \rtimes S_3$. It turns out that (M_1, M_2, M_3) forms the triplet $\mathbf{3}_{1(1)}$ of the $\Delta(54)$ group.

Next, we read off the representation of the other matter fields under the $\Delta(54)$ group. First, the fields (M_1', M_2', M_3'), which have opposite $U(1)^2$ charges and Z_3^2 charges when compared to the M_i field, lead to the $\mathbf{3}_{1(2)}$ of $\Delta(54)$. The field M can be regarded as the trivial singlet $\mathbf{1}_+$ of $\Delta(54)$. In the case of the fields (N_1, N_2, N_3), we use the linear combinations $\tilde{N}_1 \equiv (N_1 + N_2 + N_3)/\sqrt{3}$, $\tilde{N}_2 \equiv (N_1 + \omega N_2 + \omega^2 N_3)/\sqrt{3}$ and $\tilde{N}_3 \equiv (N_1 + \omega^2 N_2 + \omega N_3)/\sqrt{3}$. In this basis, one sees that they transform as a $\mathbf{1} \oplus \mathbf{2}$ of the S_3 group. After the VEV, these fields have trivial Z_3^2 charges, so they correspond to $\mathbf{1}_+ \oplus \mathbf{2}_1$ of the $\Delta(54)$ group. Note, that instead of the M_i which have $U(1)^2$ charges $e_i/3$, we can also introduce fields with charges $-2e_i/3$. Since the Z_3^2 charges of such fields are identical to the M_i, they also lead to the $\mathbf{3}_{1(1)}$ representation. As the result, we can realize $\mathbf{1}_+, \mathbf{1}_+ \oplus \mathbf{2}_1, \mathbf{3}_{1(1)}, \mathbf{3}_{1(2)}$ representations of the $\Delta(54)$ group in our setup.

2.5 A_4 group

We consider a $U(1)^2 \rtimes Z_3$ model with the field contents as in Table 5. There, we add fields A_i to the field contents of the model for the S_4 group (Table 3). We define the two-dimensional $U(1)^2$ charges as

$$w_1 \equiv (\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}),\ w_2 \equiv (-\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}),\ w_3 \equiv (0, -\frac{\sqrt{6}}{3}). \tag{15}$$

The introduction of A_i fields breaks the original S_3 symmetry to a Z_3 symmetry (under reflections, their $U(1)^2$ charges are not mapped onto each other). Then, this model has a

Field	$U(1)^2$ charge	Z_2^2 charge	A_4 rep.
U_1, U_2, U_3	$-e_1, -e_2, -e_3$	$(0,0),(0,0),(0,0)$	—
M_1, M_2, M_3	$\frac{e_1}{2}, \frac{e_2}{2}, \frac{e_3}{2}$	$(1,1),(1,0),(0,1)$	$\mathbf{3}$
M	0	$(0,0)$	$\mathbf{1}$
N_1, N_2, N_3	e_1, e_2, e_3	$(0,0),(0,0),(0,0)$	$\mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{1}''$
A_1, A_2, A_3	$\frac{3w_1}{2}, \frac{3w_2}{2}, \frac{3w_3}{2}$	$(1,0),(0,1),(1,1)$	$\mathbf{3}$

Table 5: Field contents of the $U(1)^2 \rtimes Z_3$ model for the A_4 group. Besides the $U(1)^2$ charges, the charges under the unbroken discrete Z_2^2 subgroup of $U(1)^2$ are shown. Representations under the resulting A_4 group are also shown.

$U(1)^2 \rtimes Z_3$ structure, the Z_3 symmetry acting as $U_1 \to U_2 \to U_3 \to U_1$ and $M_1 \to M_2 \to M_3 \to M_1$, etc. We consider a VEV relation as

$$\langle U_1 \rangle = \langle U_2 \rangle = \langle U_3 \rangle. \tag{16}$$

This VEV relation maintains Z_3,

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{17}$$

but breaks $U(1)^2$ to its Abelian subgroup Z_2^2. The two Z_2 charges z_1, z_2 in Table 5 are determined by $z_1 = 2(q_1/\sqrt{2} - q_2/\sqrt{6}) \pmod 2$ and $z_2 = 2(q_1/\sqrt{2} + q_2/\sqrt{6}) \pmod 2$, and the Z_2^2 action is given by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{18}$$

By combining (17) and (18), this leads to non-Abelian discrete symmetry $A_4 \cong (Z_2 \times Z_2) \rtimes Z_3$. It turns out that (M_1, M_2, M_3) forms the triplet $\mathbf{3}$ of the A_4 group.

Next, we read off the A_4 representation of the other fields. First the field M can be regarded as the trivial singlet $\mathbf{1}$ of A_4. The fields (A_1, A_2, A_3) have a similar structure to the fields M_i, and they also lead to a $\mathbf{3}$ of A_4. In the case of the fields (N_1, N_2, N_3), we use the linear combinations $\tilde{N}_1 \equiv (N_1 + N_2 + N_3)/\sqrt{3}, \tilde{N}_2 \equiv (N_1 + \omega N_2 + \omega^2 N_3)/\sqrt{3}$ and $\tilde{N}_3 \equiv (N_1 + \omega^2 N_2 + \omega N_3)/\sqrt{3}$. In this basis, the three fields transform as $\mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{1}''$ of Z_3. After the VEV, these fields have trivial Z_2^2 charges, so they correspond to $\mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{1}''$ of the A_4 group. Note that other fields (M_1', M_2', M_3') with $U(1)^2$ charges $(2n+1)e_i/2$, where n is an integer, also lead to $\mathbf{3}$ representation since they have same Z_2^2 charges as M_i. As a result, we can realize $\mathbf{1}, \mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{1}'', \mathbf{3}$ representations of A_4 in this setup.

2.6 $\Delta(27)$ group

We consider a $U(1)^2 \rtimes Z_3$ model with the field contents as in Table 6. There, we have added fields A_i and B_i to the field content of the $\Delta(54)$ model (Table 4). These fields break the S_3

Field	$U(1)^2$ charge	Z_3^2 charge	$\Delta(27)$ rep.
U_1, U_2, U_3	$-e_1, -e_2, -e_3$	$(0,0), (0,0), (0,0)$	—
M_1, M_2, M_3	$\frac{e_1}{3}, \frac{e_2}{3}, \frac{e_3}{3}$	$(1,1), (2,0), (0,2)$	$\mathbf{3}_{[0][1]}$
M_1', M_2', M_3'	$-\frac{e_1}{3}, -\frac{e_2}{3}, -\frac{e_3}{3}$	$(2,2), (1,0), (0,1)$	$\mathbf{3}_{[0][2]}$
M	0	$(0,0)$	$\mathbf{1}_{0,0}$
N_1, N_2, N_3	e_1, e_2, e_3	$(0,0), (0,0), (0,0)$	$\mathbf{1}_{0,0} \oplus \mathbf{1}_{1,0} \oplus \mathbf{1}_{2,0}$
A_1, A_2, A_3	w_1, w_2, w_3	$(1,2), (1,2), (1,2)$	$\mathbf{1}_{0,2} \oplus \mathbf{1}_{1,2} \oplus \mathbf{1}_{2,2}$
B_1, B_2, B_3	$2w_1, 2w_2, 2w_3$	$(2,1), (2,1), (2,1)$	$\mathbf{1}_{0,1} \oplus \mathbf{1}_{1,1} \oplus \mathbf{1}_{2,1}$

Table 6: Field contents of the $U(1)^2 \rtimes Z_3$ model for the $\Delta(27)$ group. Besides the $U(1)^2$ charges, the charges under the unbroken discrete Z_3^2 subgroup of $U(1)^2$ are shown. Representations under the resulting $\Delta(27)$ group are also shown.

symmetry to a Z_3 symmetry. We now consider the VEV relation (16), which maintains Z_3 (17) but breaks $U(1)^2$ to its Abelian subgroup Z_3^2. The two Z_3 charges z_1, z_2 in Table 6 are determined as $z_1 = 3(q_1/\sqrt{2} - q_2/\sqrt{6})$ (mod 3) and $z_2 = 3(q_1/\sqrt{2} + q_2/\sqrt{6})$ (mod 3). Also, the Z_3^2 action is given by

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^{-1} \end{pmatrix}. \tag{19}$$

The generators (17) and (19) generate a non-Abelian discrete symmetry $\Delta(27) \cong (Z_3 \times Z_3) \rtimes Z_3$. It turns out that (M_1, M_2, M_3) forms the triplet $\mathbf{3}_{[0][1]}$ of the $\Delta(27)$ group.

Next we read off the representation of the other matter fields under the $\Delta(27)$ group. First, the fields (M_1', M_2', M_3') which have opposite $U(1)^2$ charges and Z_3^2 charges when compared to the fields M_i lead to a $\mathbf{3}_{[0][2]}$ of the $\Delta(27)$ group. The field M can be regarded as the trivial singlet $\mathbf{1}_{0,0}$ of $\Delta(27)$. In the case of the fields (N_1, N_2, N_3), we use the linear combinations $\tilde{N}_1 \equiv (N_1 + N_2 + N_3)/\sqrt{3}$, $\tilde{N}_2 \equiv (N_1 + \omega N_2 + \omega^2 N_3)/\sqrt{3}$ and $\tilde{N}_3 \equiv (N_1 + \omega^2 N_2 + \omega N_3)/\sqrt{3}$. In this basis, the three fields transform as $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}''$ of Z_3. After the VEV, these fields have the trivial Z_3^2 charges, so they correspond to $\mathbf{1}_{0,0} \oplus \mathbf{1}_{1,0} \oplus \mathbf{1}_{2,0}$ of the $\Delta(27)$ group. Next we consider the fields (A_1, A_2, A_3). They have degenerate Z_3^2 charges, so by diagonalization we observe that they transform as $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}''$ under Z_3. Then, these fields lead to a $\mathbf{1}_{0,2} \oplus \mathbf{1}_{1,2} \oplus \mathbf{1}_{2,2}$ of the $\Delta(27)$ group. Similarly (B_1, B_2, B_3) lead to $\mathbf{1}_{0,1} \oplus \mathbf{1}_{1,1} \oplus \mathbf{1}_{2,1}$ of the $\Delta(27)$ group. As a result, we can realize the $\mathbf{1}_{0,0}, \mathbf{1}_{0,0} \oplus \mathbf{1}_{1,0} \oplus \mathbf{1}_{2,0}, \mathbf{1}_{0,1} \oplus \mathbf{1}_{1,1} \oplus \mathbf{1}_{2,1}, \mathbf{1}_{0,2} \oplus \mathbf{1}_{1,2} \oplus \mathbf{1}_{2,2}, \mathbf{3}_{[0][1]}, \mathbf{3}_{[0][2]}$ representations of the $\Delta(27)$ group in this setup.

3 $U(1)^2 \rtimes S_3$ lepton flavor model

In this section we present a concrete model for the lepton sector based on the $U(1)^2 \rtimes S_3$ symmetry, which is related to the $\Delta(54)$ discrete symmetry discussed in Section 2.4. Several

Field	$U(1)^2$ charge	Z_2 charge	$\Delta(54)$ rep.
(L_e, L_μ, L_τ)	$(\frac{2e_1}{3}, \frac{2e_2}{3}, \frac{2e_3}{3})$	0	$\mathbf{3}_{1(2)}$
(e^c, μ^c, τ^c)	$(-3e_1, -3e_2, -3e_3)$	1	$\mathbf{1}_+ \oplus \mathbf{2}_1$
H_u	0	0	$\mathbf{1}_+$
H_d	0	0	$\mathbf{1}_+$
(A_1, A_2, A_3)	$(\frac{2e_1}{3}, \frac{2e_2}{3}, \frac{2e_3}{3})$	0	$\mathbf{3}_{1(2)}$
(B_1, B_2, B_3)	$(-\frac{4e_1}{3}, -\frac{4e_2}{3}, -\frac{4e_3}{3})$	0	$\mathbf{3}_{1(2)}$
(C_1, C_2, C_3)	$(\frac{e_1}{3}, \frac{e_2}{3}, \frac{e_3}{3})$	0	$\mathbf{3}_{1(1)}$
(D_1, D_2, D_3)	$(\frac{7e_1}{3}, \frac{7e_2}{3}, \frac{7e_3}{3})$	1	$\mathbf{3}_{1(1)}$

Table 7: Field contents of the $U(1)^2 \rtimes S_3 \times Z_2$ lepton flavor model. $U(1)^2$ charges and Z_2 charges are shown. Representations under the $\Delta(54)$ group are also shown.

interesting flavor models based on the $\Delta(54)$ symmetry have been investigated in [22, 23, 24, 25, 26].

Here we consider a supersymmetric model with $U(1)^2 \rtimes S_3 \times Z_2$ symmetry, and with the field content as in Table 7. There, in addition to the MSSM fields (the lepton doublets (L_e, L_μ, L_τ), the right-handed lepton fields (e^c, μ^c, τ^c) and Higgs doublet pairs (H_u, H_d)) we introduce flavon fields A_i, B_i, C_i and D_i. The VEV of the flavon fields breaks the $U(1)^2 \rtimes S_3$ symmetry completely. Corresponding representations under $\Delta(54)$ are also shown in Table 7. It is also possible to add other flavon fields, e.g. fields U_i in Table 4, and consider the situation where the VEV of the fields, $\langle U_1 \rangle = \langle U_2 \rangle = \langle U_3 \rangle$, breaks the symmetry as $U(1)^2 \rtimes S_3 \to \Delta(54)$ at an intermediate scale. In this paper we do not consider this possibility.

3.1 Yukawa mass matrices

First, we consider the Yukawa sector of the model. By invariance under $U(1)^2 \rtimes S_3 \times Z_2$, the superpotentials of the neutrino sector and the charged lepton sector are given by

$$\begin{aligned} W_\nu = {} & y_1^\nu \left(B_1 L_e L_e + B_2 L_\mu L_\mu + B_3 L_\tau L_\tau\right) H_u H_u/\Lambda^2 \\ & + y_2^\nu \left(A_1(L_\mu L_\tau + L_\tau L_\mu) + A_2(L_e L_\tau + L_\tau L_e) + A_3(L_e L_\mu + L_\mu L_e)\right) H_u H_u/\Lambda^2 \\ & + y_3^\nu \left(C_1^2(L_\mu L_\tau + L_\tau L_\mu) + C_2^2(L_e L_\tau + L_\tau L_e) + C_3^2(L_e L_\mu + L_\mu L_e)\right) H_u H_u/\Lambda^3, \end{aligned} \tag{20}$$

and

$$W_e = y_1^e \left(D_1 L_e e^c + D_2 L_\mu \mu^c + D_3 L_\tau \tau^c\right) H_d/\Lambda, \tag{21}$$

respectively. Here, we assume a UV cutoff scale Λ. Then the mass matrices are given by

$$M_\nu = \frac{v_u^2}{\Lambda^2}\begin{pmatrix} y_1^\nu b_1 & y_2^\nu a_3 & y_2^\nu a_2 \\ y_2^\nu a_3 & y_1^\nu b_2 & y_2^\nu a_1 \\ y_2^\nu a_2 & y_2^\nu a_1 & y_1^\nu b_3 \end{pmatrix} + \frac{y_3^\nu v_u^2}{\Lambda^3}\begin{pmatrix} 0 & c_3^2 & c_2^2 \\ c_3^2 & 0 & c_1^2 \\ c_2^2 & c_1^2 & 0 \end{pmatrix}, \tag{22}$$

$$M_e = \frac{y_1^e v_d}{\Lambda}\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}, \tag{23}$$

where we used the following definition for the VEVs of the flavon fields:

$$\langle (A_1, A_2, A_3) \rangle = (a_1, a_2, a_3), \tag{24}$$

$$\langle (B_1, B_2, B_3) \rangle = (b_1, b_2, b_3), \tag{25}$$

$$\langle (C_1, C_2, C_3) \rangle = (c_1, c_2, c_3). \tag{26}$$

$$\langle (D_1, D_2, D_3) \rangle = (d_1, d_2, d_3). \tag{27}$$

Note that the charged lepton mass matrix is diagonal. Thus, the mixing angles are determined only by the neutrino mass matrix.

3.2 Flavon potential and vacuum alignment

Next we consider the flavon sector. The superpotential up to three-point level including only flavon fields is given by

$$W_f = \lambda_1 A_1 A_2 A_3 + \lambda_2 B_1 B_2 B_3 + \lambda_3 C_1 C_2 C_3 + \lambda_4 \left(A_1^2 B_1 + A_2^2 B_2 + A_3^2 B_3 \right). \tag{28}$$

The F-flatness condition for the flavon superpotential leads to (for $i \neq j \neq k \neq i$)

$$0 = \frac{\partial W_f}{\partial A_k} = \lambda_1 A_i A_j + 2\lambda_4 A_k B_k, \tag{29}$$

$$0 = \frac{\partial W_f}{\partial B_k} = \lambda_2 B_i B_j + \lambda_4 A_k^2, \tag{30}$$

$$0 = \frac{\partial W_f}{\partial C_k} = \lambda_3 C_i C_j, \tag{31}$$

$$0 = \frac{\partial W_f}{\partial D_k} = 0. \tag{32}$$

There are two branches of solutions:

(a) Let us first assume $A_i \neq 0$ and $B_i \neq 0$. Then we can solve (29) for B_k and insert the solution in to (30). Then, we obtain the condition $4\lambda_4^3 = -\lambda_2 \lambda_1^2$, so we can choose the VEVs as:

$$\langle A_i \rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \langle B_i \rangle = -\frac{\lambda_1}{2\lambda_4}\begin{pmatrix} \frac{a_2 a_3}{a_1} \\ \frac{a_3 a_1}{a_2} \\ \frac{a_1 a_2}{a_3} \end{pmatrix}. \tag{33}$$

(b) If not all $A_i \neq 0$ or $B_i \neq 0$ then there exist solutions, and they can be brought into the following form by an S_3 transformation:

$$\langle A_i \rangle = \begin{pmatrix} 0 \\ 0 \\ a_3 \end{pmatrix}, \quad \langle B_i \rangle = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix}, \tag{34}$$

with the condition $\lambda_2 b_1 b_2 + \lambda_4 a_3^2 = 0$.

Furthermore, the VEVs of any two components C_i must be zero. In the following we assume

$$\langle C_i \rangle = \begin{pmatrix} c_1 \\ 0 \\ 0 \end{pmatrix}. \tag{35}$$

The D_i are not constrained from F-flatness.

3.3 Neutrino mass/mixing properties

In the following we consider only the case (a). By inserting the VEVs the mass matrix becomes

$$M_\nu = \frac{v_u^2}{\Lambda^2} \begin{pmatrix} -y_1^\nu \frac{\lambda_1}{2\lambda_4} \frac{a_2 a_3}{a_1} & y_2^\nu a_3 & y_2^\nu a_2 \\ y_2^\nu a_3 & -y_1^\nu \frac{\lambda_1}{2\lambda_4} \frac{a_1 a_3}{a_2} & y_2^\nu a_1 \\ y_2^\nu a_2 & y_2^\nu a_1 & -y_1^\nu \frac{\lambda_1}{2\lambda_4} \frac{a_1 a_2}{a_3} \end{pmatrix} + \frac{y_3^\nu v_u^2}{\Lambda^3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_1^2 \\ 0 & c_1^2 & 0 \end{pmatrix}. \tag{36}$$

For the later convenience we define the following parameters

$$a_2' \equiv \frac{a_2}{a_1}, \; a_3' \equiv \frac{a_3}{a_1}, \; A \equiv \frac{v_u^2 y_2^\nu a_1}{\Lambda^2}, \; B \equiv -\frac{y_1^\nu}{y_2^\nu} \frac{\lambda_1}{2\lambda_4}, \; C \equiv \frac{y_3^\nu}{y_2^\nu} \frac{c_1^2}{a_1 \Lambda}, \tag{37}$$

(A, B and C not to be confused with the flavon fields A_i, B_i and C_i) and rewrite the mass matrix (36) as

$$M_\nu = A \begin{pmatrix} B a_2' a_3' & a_3' & a_2' \\ a_3' & B \frac{a_3'}{a_2'} & 1 + C \\ a_2' & 1 + C & B \frac{a_2'}{a_3'} \end{pmatrix}. \tag{38}$$

It turns out that this mass matrix has the following relations,

$$\frac{M_{22}}{M_{11}} = \left(\frac{M_{23} - AC}{M_{13}} \right)^2, \tag{39}$$

$$\frac{M_{33}}{M_{22}} = \left(\frac{M_{13}}{M_{12}} \right)^2, \tag{40}$$

$$\frac{M_{11}}{M_{33}} = \left(\frac{M_{12}}{M_{23} - AC} \right)^2. \tag{41}$$

Note that the three equations are dependent. Actually, the third equation is a consequence of the first and the second equations. The first equation (39) can be solved by AC as

$$AC = M_{23} \pm M_{13}\sqrt{\frac{M_{22}}{M_{11}}}, \tag{42}$$

thus if the mass matrix M_ν is fixed, the parameter AC can be derived. Hence, (40) is a prediction for ratios of elements of the neutrino mass matrix M_ν.

Now, we investigate whether this model can explain the experimental values of mass hierarchies and mixings. In our model, the charged lepton mass matrix (23) already takes a diagonal form, so the PMNS mixing matrix U_{PMNS} is given by a unitary matrix U_ν which diagonalizes the neutrino mass matrix (38) as

$$U_{\text{PMNS}} = U_\nu = R_{23}U_{13}R_{12}P_{12}. \tag{43}$$

Here, the rotation matrices are defined by three mixing angles $(\theta_{12}, \theta_{23}, \theta_{13})$ and three CP phases $(\delta, \beta_1, \beta_2)$ as

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix}, \; U_{13} = \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13}e^{i\delta} & 0 & \cos\theta_{13} \end{pmatrix}, \tag{44}$$

$$R_{12} = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; P_{12} = \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{45}$$

For simplicity, here we consider only the case where

$$\delta = \beta_1 = \beta_2 = 0. \tag{46}$$

We also set the mixing angle θ_{12} to fit the experimental value as

$$\theta_{12} = 35.3^\circ. \tag{47}$$

Then the mixing matrix (43) is a real matrix. As for the neutrino mass differences, we wish to reproduce the case of the inverted hierarchy:

$$\Delta m_{21}^2 = m_2^2 - m_1^2 = 7.60 \times 10^{-5}\ \text{eV}^2, \tag{48}$$

$$\Delta m_{31}^2 = m_3^2 - m_1^2 = -2.38 \times 10^{-3}\ \text{eV}^2, \tag{49}$$

and regard the third family neutrino mass m_3 as a parameter. These values are consistent with the global analysis in [27] within 2σ range. The neutrino mass matrix is then obtained as

$$M_\nu = U_{\text{PMNS}}\ MU_{\text{PMNS}}^T, \tag{50}$$

where $M = \text{diag}(m_1, m_2, m_3)$. In Figure 1, we show a prediction for various values of $(m_3, \theta_{13}, \theta_{23})$ from the ratio condition of this mass matrix (40). In the figure we show solutions of the mixing angle θ_{23} against the third generation neutrino mass m_3 for (40) with fixed θ_{13} angles, $\theta_{13} = 8.2^\circ, 8.7^\circ, 9.1^\circ$, which is in 2σ range.

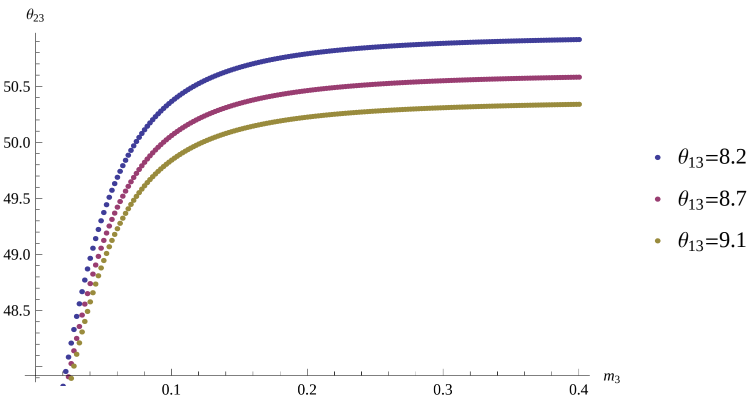

Figure 1: The mixing angle θ_{23} (in degrees) against the third generation neutrino mass m_3 (in eV) for various values of the mixing angle θ_{13} (in degrees).

Actually, there exist solutions for our parameters (A, B, C, a_2', a_3') in (38) to realize these experimental values. For example, if we take the parameters to be

$$\begin{aligned} A &= 0.00198\,\mathrm{eV}, \\ B &= 30.5, \\ C &= -5.94, \\ a_2' &= -1.09, \\ a_3' &= -1.06, \end{aligned} \tag{51}$$

we can obtain $m_3 = 0.05\,\mathrm{eV}, \theta_{13} = 8.7°, \theta_{23} = 49.1°$. This solution is also consistent with the 2σ range of recent fits from neutrinoless double beta decay [28]:

$$m_{\beta\beta} \approx 0.05\,\mathrm{eV}, \tag{52}$$

$$\Sigma = m_1 + m_2 + m_3 \approx 0.15\,\mathrm{eV}. \tag{53}$$

3.4 Charged lepton masses

Next, we consider the charged lepton mass matrix (23). We want to fix the charged lepton masses as

$$m_e = \frac{y_l^e v_d}{\Lambda} \cdot d_1 = 0.5 \times 10^6 \text{ eV}, \tag{54}$$

$$m_\mu = \frac{y_l^e v_d}{\Lambda} \cdot d_2 = 105 \times 10^6 \text{ eV}, \tag{55}$$

$$m_\tau = \frac{y_l^e v_d}{\Lambda} \cdot d_3 = 1776 \times 10^6 \text{ eV}. \tag{56}$$

The charged lepton masses are constrained from the D-flatness condition, which for this model is given by

$$\begin{aligned}&\frac{7}{3}e_1|d_1|^2+\frac{7}{3}e_2|d_2|^2+\frac{7}{3}e_3|d_3|^2-\frac{4}{3}e_1|b_1|^2-\frac{4}{3}e_2|b_2|^2-\frac{4}{3}e_3|b_3|^2\\&+\frac{2}{3}e_1|a_1|^2+\frac{2}{3}e_2|a_2|^2+\frac{2}{3}e_3|a_3|^2+\frac{1}{3}e_1|c_1|^2=0,\end{aligned}\tag{57}$$

or equivalently

$$\begin{aligned}&+\frac{7}{3}\left(e_1|m_e|^2+e_2|m_\mu|^2+e_3|m_\tau|^2\right)\cdot\left|\frac{\Lambda}{y_1^e v_d}\right|^2\\&-\frac{4}{3}\left(e_1|a_2'a_3'A|^2+e_2\left|\frac{Aa_3'}{a_2'}\right|^2+e_3\left|\frac{Aa_2'}{a_3'}\right|^2\right)\cdot\left|\frac{\lambda_1\Lambda^2}{2\lambda_4 v_u^2 y_2^\nu}\right|^2\\&+\frac{2}{3}\left(e_1|A|^2+e_2|a_2'A|^2+e_3|a_3'A|^2\right)\cdot\left|\frac{\Lambda^2}{v_u^2 y_2^\nu}\right|^2+\frac{1}{3}e_1|AC|\cdot\left|\frac{\Lambda^3}{y_3^\nu v_u^2}\right|=0.\end{aligned}\tag{58}$$

After inserting the solution (A,B,C,a_2',a_3') from (51) we can numerically solve (58) as a linear equation. Here, we only consider the simplified case where $|\lambda_1/(2\lambda_4)|=1$. Then, we obtain a single solution,

$$\left|\frac{y_1^e y_2^\nu}{y_3^\nu}v_d\right|\approx 2.10\,\mathrm{GeV},\tag{59}$$

$$\left|\frac{(y_1^e)^2}{y_3^\nu}\frac{v_d^2}{v_u^2}\Lambda\right|\approx 7.06\times 10^{12}\ \ \mathrm{GeV}.\tag{60}$$

Then, by taking "natural" values, $|y_1^e|=|y_2^\nu|=|y_3^\nu|=1$, and by imposing

$$v_u^2+v_d^2=(173\,\mathrm{GeV})^2\tag{61}$$

we arrive at

$$\tan\beta=\frac{v_u}{v_d}\approx 82.4,\tag{62}$$

$$\Lambda\approx 4.79\times 10^{16}\ \ \mathrm{GeV}.\tag{63}$$

Other values of $\tan\beta$ and Λ are possible by appropriately adjusting the couplings.

4 Conclusion

In this work, motivated by a gauge origin of discrete symmetries in the framework of the heterotic orbifold models, we have investigated gauge theoretical realizations of non-Abelian discrete flavor symmetries. We have shown that phenomenologically interesting discrete symmetries are realized effectively from a $U(1)^n \rtimes S_m$ or $U(1)^n \rtimes Z_m$ gauge theory. These theories

can be regarded as UV completions of discrete flavor models. The main difference between a discrete flavor model and a $U(1)$ flavor model as shown in this paper can be seen in the field interactions. Namely, some fields in a discrete flavor model can be distinguished in a $U(1)$ flavor model. For example, the $\mathbf{3}_{1(1)}$ representation field of the $\Delta(54)$ symmetry can be described by several $U(1)^2$ charges, $(e_1/3, e_2/3, e_3/3)$, $(-2e_1/3, -2e_2/3, -2e_3/3)$ etc. Thus a superpotential in a $U(1)$ flavor model can be different from the one of the corresponding discrete flavor model. In general, $U(1)^n \rtimes S_m$ and $U(1)^n \rtimes Z_m$ flavor models are constrained more than flavor models with non-Abelian discrete flavor symmetries, which are subgroups of $U(1)^n \rtimes S_m$ and $U(1)^n \rtimes Z_m$, because symmetries are larger. Our results would provide a new insight on flavor models.

We have introduced the specific combination of $U(1)^2$ charges, e_1, e_2, and e_3, to realize S_4, $\Delta(54)$, A_4 and $\Delta(27)$. They correspond to weights of the triplet (or anti-triplet) representation of $SU(3)$. In fact, $U(1)^2 \rtimes S_3$ is a subgroup of $SU(3)$, where S_3 is associated with the Weyl group. We also obtained genuine $U(1)^2 \rtimes S_3$ representations which are not obtained from $SU(3)$ triplets by spontaneous symmetry breaking. Also, in a stringy realization of $\Delta(54)$, the $SU(3)$ gauge symmetry appears in toroidal compactification, and the non-zero roots can be projected out by an orbifold projection [20]. This may also suggest that a similar situation can be realized field-theoretically in a higher-dimensional $SU(3)$ gauge theory with a suitable orbifold boundary condition.

Anomalies of non-Abelian discrete symmetries are important [29]. Anomalous discrete symmetries would be violated by non-perturbative effects, but its breaking effects might be small depending on dynamical scales of non-perturbative effects. By our construction, discrete Abelian symmetries originating from $U(1)^n$ of $U(1)^n \rtimes S_m$ and $U(1)^n \rtimes Z_m$ are always anomaly-free and exact symmetries, but S_m and Z_m of $U(1)^n \rtimes S_m$ and $U(1)^n \rtimes Z_m$ can include anomalous discrete symmetries depending on the model.

We have constructed a concrete flavor model for the lepton sector based on the $U(1)^2 \rtimes S_3$ continuous gauge theory. We have shown that it is possible obtain a realistic flavor structure from this model. Since the model is based on an extended symmetry the number of the parameters is relatively few. In particular, we could show a relation between the angle θ_{23} and third generation neutrino mass m_3.

We have shown six types of gauge realizations of non-Abelian discrete symmetries. However, further extensions are possible. For example, extensions to higher N, $\Delta(6N^2)$, is possible if we consider models with $U(1)$ charges $q = e_i/N$. It is also possible to include further representations of e.g. $U(1)^2 \rtimes S_3$ which we did not cover here for the sake of simplicity. The general representation theory of these semidirect groups is obtained from the little group method of Wigner, which is familiar from the representation theory of the Poincaré group. Then, e.g. in the case of $U(1)^2 \rtimes S_3$ one obtains an uncharged singlet representation which transforms as $\mathbf{1}'$ under S_3 while being uncharged under the $U(1)^2$.

A phenomenological implication of our $U(1)$ flavor models is that there should be Z' boson(s) which originate from $U(1)$ gauge groups in the effective theory. In this framework Z' bosons and flavor structures are related. Since we assigned different $U(1)$ charges to the three-generation leptons, the Z' bosons have flavor dependent interactions. Thus, if Z' bosons

are light as e.g. the TeV scale, they can be a probe of the flavor structure. It will be interesting to investigate Z' phenomenology by extending well-known discrete flavor models.

Acknowledgement

S.K. wishes to thank Otto C.W. Kong for helpful discussions. F.B. was supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports, and Culture (MEXT), Japan (No. 23104011). T.K. was supported in part by the Grant-in-Aid for Scientific Research No. 25400252 from the Ministry of Education, Culture, Sports, Science and Technology of Japan. S.K. was supported by the Taiwan's National Science Council under grant NSC102-2811-M-033-008.

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